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# TIME SERIES ANALYSIS USING FRACTAL THEORY

***Abstract***. *Financial markets have become a center of attention and a field of study for over a century, especially nowadays when there is an abundance of data about them, however, behavior of financial markets still stays as mystery and a field that is rife with unpredictability and uncertainty. There are many methods and techniques for analyzing time series data. Due to the fact that prices are formed by decisions made by millions of people across the planet the price in itself holds the reflection of all of those decisions which itself is a natural process. Therefore, we can look at the financial data as a fractal object and analyze them. Method that has been primarily used to analyze time series is fractal dimension which we can use to measure of the time series, in other words, we can measure how correlated is the time series and thus have an opinion about its future behavior. As we later find out, fractal dimension provides interesting insights into the time series and most importantly, it is a method that can analyze pictures and thus it is not restricted by data formats.*

***Keywords****: Uncertainty, Time series, Fractals, Fractal Dimension, Autocorrelation, Hurst Exponent*

***JEL classification:*** *C1, C22, C32, D81*

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# დროითი მწკრივების ანალიზი ფრაქტალური თეორიის გამოყენებით

***აბსტრაქტი.*** *ფინანსური ბაზრები საუკუნეზე მეტი ხნის განმავლობაში მოექცა ყურადღების ცენტრში და გახდა კვლევის სფერო, განსაკუთრებით დღეს, როდესაც მათ შესახებ უამრავი მონაცემი არსებობს, თუმცა ფინანსური ბაზრების ქცევა კვლავ საიდუმლოდ რჩება და არაპროგნოზირებადობითა და განუზღვრელობით ხასიათდება. არსებობს მრავალი მეთოდი და ხერხი დროთი მწკრივების მონაცემების გასაანალიზებლად. ფინანსურ ბაზრებზე ფასები ფორმირდება მილიონობით ადამიანის მიერ მთელს პლანეტაზე მიღებული გადაწყვეტილებებით და თავისთავად წარმოადგენს ამ გადაწყვეტილებების ერთგავრ ანარეკლს, ეს კი თავისთავად ბუნებრივი პროცესია. აქედან გამომდინარე, ჩვენ შეგვიძლია განვიხილოთ ფინანსური მონაცემები, როგორც ფრაქტალური ობიექტები და გავაანალიზოთ ისინი. მეთოდი, რომელიც ძირითადად გამოვიყენებთ დროითი მწკრივების გასაანალიზებლად, არის ფრაქტალური განზომილება, რომელიც, შეგვიძლია გამოვიყენოთ დროითი მწკრივების გასაზომად, სხვა სიტყვებით რომ ვთქვათ, შეგვიძლია გავზომოთ რამდენად კორელირებულია დროითი მწკრივი და ამით გვქონდეს მოსაზრება მის მომავალ ქცევაზე. ფრაქტალური განზომილება გვაძლევს*

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***JEL კლასიფიკაცია:** C1, C22, C32, D81*

**Introduction and review of literature**

In the conditions of modern uncertainty (Bedianashvili, 2023) the creation of confrontational global factors (Papava, 2022) gives economic processes a high degree of complexity, which requires the use of new tools when considering the dynamic aspect. This problem is particularly apparent in relation to economic time series of a different nature.

Times series analysis has become more and more popular after development of financial markets which always requires up to date information on the future behaviors of financial products encompassing exchange rates, options, futures, stock prices, crypto currencies and so on. Some aspects of this problem in various works.

The data about these time series can exist in different forms and formats such as Excel files, text or pictures. Having data in excel format makes it very easy to analyze, however, not all the data is preserved in such a format. Fractal analysis of time series allows analysis of pictures which makes it very useful when data is only available in picture formats. It is almost impossible to analyze such time series with traditional methods. Conducted analysis show that fractal theory can provide an estimate for correlation in time series and can be used to analyze data that is only available in picture format.

In this study, time series has been transformed into fractal objects and further analysis is done with Hurst's exponent. Furthermore, autocorrelation function (ACF) is applied to the analyzed data and serves as a well trusted and approved method for evaluating relationships in data.

The first time when Mathematics was used to analyze financial markets dates back to year 1900 when Louis Bachelier used mathematics for analyzing government bonds and other financial data (Mandelbrot, 2006). Fractals have been highly introduced by Benoit Mandelbrot in analysis of financial data. According to Mandelbrot "Price movements do not follow the well-mannered bell curve assumed by modern finance" and we can easily observe that by looking at time series data of exchange rates, option prices, cryptocurrencies, etc. Method that has been primarily used to analyze time series is fractal dimension which, as Mandelbrot says, is a numerical measure of the "roughness" of an object. We can use this concept to measure the "roughness" of the time series, in other words, we can measure how correlated is the time series and thus have an opinion about its future behavior. Despite that, Harold Edwin Hurst suggested a method cold Rescaled Range Analysis to analyze long term correlation in time series data. Hurst's exponent that was invented by Harold Edwin Hurst can be used in combination with fractal dimension to measure autocorrelation in time series data. In this paper, both of these method have been used to estimate autocorrelation in time series data along with standard autocorrelation function to validate the results. In the field of applying fractal theory in economic research, interesting works of other authors can also be distinguished (for example, Hudson, 2006; Hurst et al, 1965; Kantelhardt, 2008; Kapecka, 2013; Lunga, 2018; Mandelbrot, 1997; 2002; 2004; McCauley et al, 2007; Mkhatvrishvili et al, 2019; Nunes et al, 2011; Pipia et al, 2020; Phrangishvili et al, 2023; Preis et al, 2011; Takayasu & Takayasu, 2011; Taylor, 2018).

**Methodology and methods**

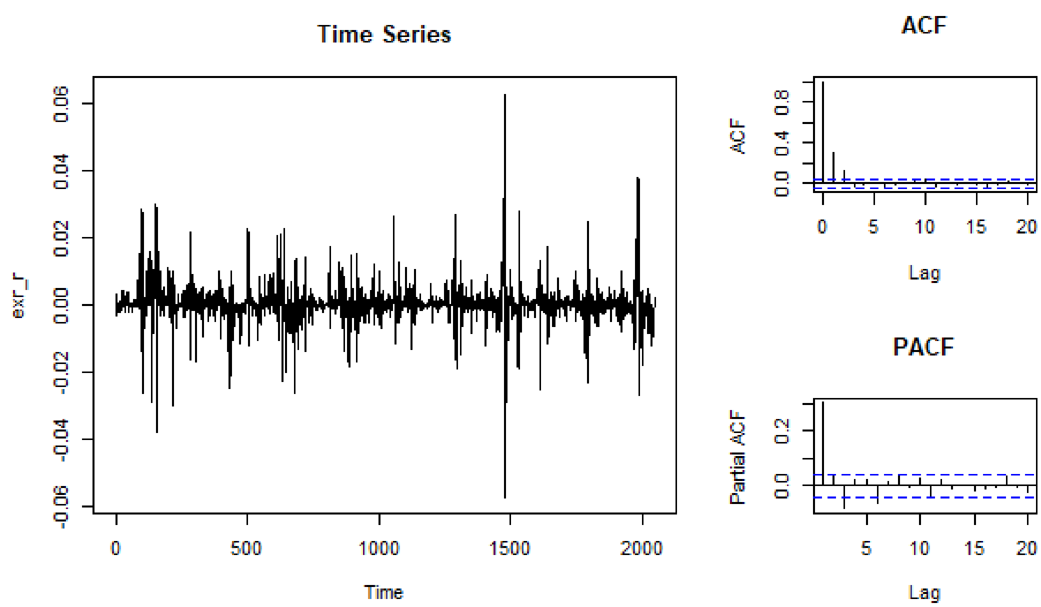
This article used general and specific research methods such as fractal theory, econometrics, analysis, synthesis, induction, deduction, scientific abstraction, and comparative analysis.

**Theoretical Generalizations, and Results**

The paper analyzes 2 time series: the exchange rate and the price of the cryptocurrency Bitcoin. Daily increment of the exchange rate of the US dollar to the GEL from July 23, 2014 to May 30, 2022 was selected to be analyzed. The behavior of the exchange rate is determined by the action of the forces of demand and supply on it, which is shaped by the decisions made by thousands of people. Let's consider the exchange rate of GEL/USD.

As can be seen from Figure 1, the analyzed time series indicate the presence of positive autocorrelation, which is obvious by the tendency of the autocorrelation function to decay step by step along with the time lag (See Figure 1). It is worth noting that the partial autocorrelation function (PACF) has only one significant value (a spike), which is significantly higher than the specified threshold value, indicating the existence of first-order autocorrelation (Chan, 2008).

**Figure 1: Autocorrelation and partial autocorrelation functions of the USD/GEL exchange rate**




*Source: National Bank of Georgia, analysis is performed in software R*

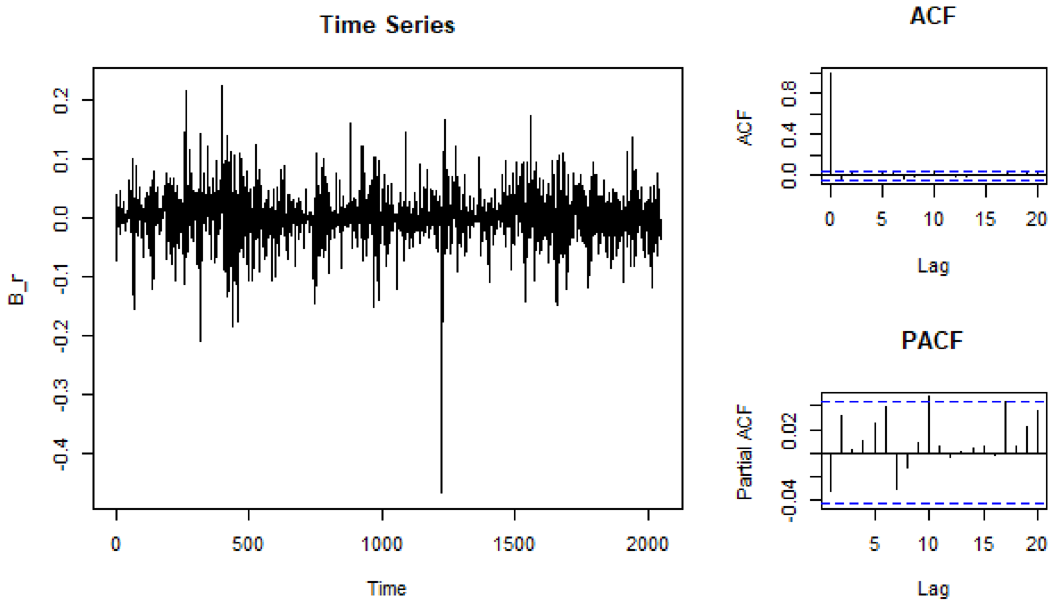
ACF is a correlation function of a time series with its lagged values. If we denote lag by k, the ACF formula can be expressed by the following equation (Ananiashvili, 2014):

$$ACF = \frac{Cov\,(u_t, u_{t-k})}{\sqrt{D(u_t)D(u_{t-k})}}$$ where $u_t$ is a time series, D - dispersion, k=1, 2, ….t

Similarly, PACF is a ACF in a time series where all the influence of any other lag is eliminated (Ananiashvili, 2014).

The second time series represents the daily price change of the cryptocurrency Bitcoin from October 31, 2016 to June 9, 2022.

**Figure 2: Bitcoin daily price change autocorrelation function and private autocorrelation function**




*Source: Yahoo Finance, analysis is performed in software R*

By looking at Figure 2, the time series of Bitcoin price change (daily increment) is not characterized by autocorrelation, ACF values do not exceed the threshold values at any lag. The same is confirmed by the partial autocorrelation function - the values on all lags are within the limits.

**Fractal dimension - Box Count method**

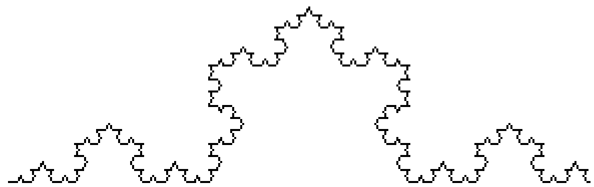
Fractal dimension in time series analysis shows how much the time series repeats itself when we change the time series observation frame.

There are several approaches to estimating the fractal dimension. The simplest approach is the so-called the method of "counting boxes", which involves dividing a time series (and any object in general) into equal-sized parts (in our case, quadrats) and then counting these quadrats.

As an illustrative example of this method, we can cite the Koch curve. The curve very much looks like a time series data although it is obtained by dividing the section into three equal parts, from which an equilateral triangle is formed in the middle, and it is repeated endlessly.

In order to move on to the use of fractal dimension, we should look at the concept of similarity factor, which is the smallest unit of which any simple figure is comprised of, for example, a section, if we take a line with length L and divide it into N equal parts, then the similarity factor will be $r(N) = 1/N$.

**Figure 3: Koch Curve**



*Source: Created using software python, Turtle library*

Similarly if we divide a square with L sides into N quadrants, then the similarity factor will be r(N) = N^(-1/2). Thus, in case of cube, the similarity factor is N^(-1/3). As dimensions increase, the similarity factor follows the logic: r(N) = N^(-1/D). If we take the logarithm of both sides of the equation (Taylor, 2018) we get the formula for fractal dimension:

$$D = -\log(N)/\log(r(N)). \qquad (1)$$

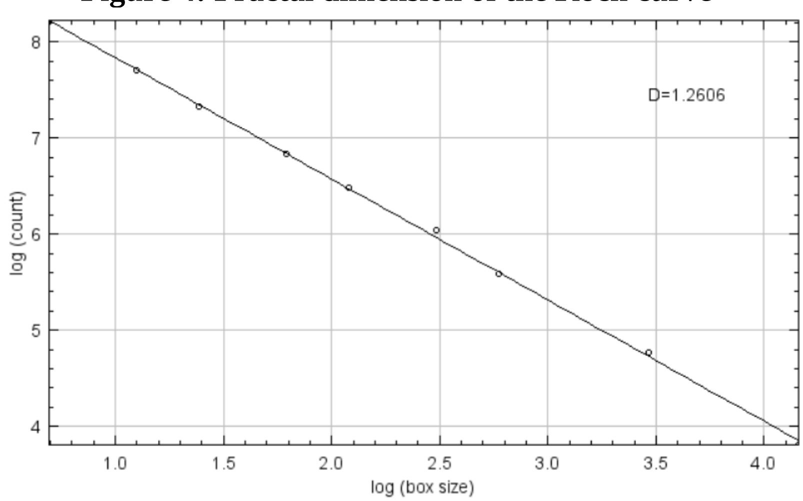
As the theory suggests, the fractal dimension for Koch Curve should be:

$$D = -\log(4)/\log(1/3) = 1.26. \quad (2)$$

Calculating the similarity dimension for figures that do not have clearly defined similar shapes is impossible with the above-mentioned approach, so in such a case the box counting method is used. In order to validate the approach, lets calculate the fractal dimension of the same curve using box counting method by which a fractal dimension is defined as a coefficient of regression between logarithmic index of the number of quadrats (log(N)) in the figure and the inverse of the size of the square side (ℓ - the side of the square) (log(1/ℓ)).

For this, we use the program ImageJ. This method later on will be used to calculate fractal dimensions for time series that we discussed in the beginning.

**Figure 4: Fractal dimension of the Koch curve**



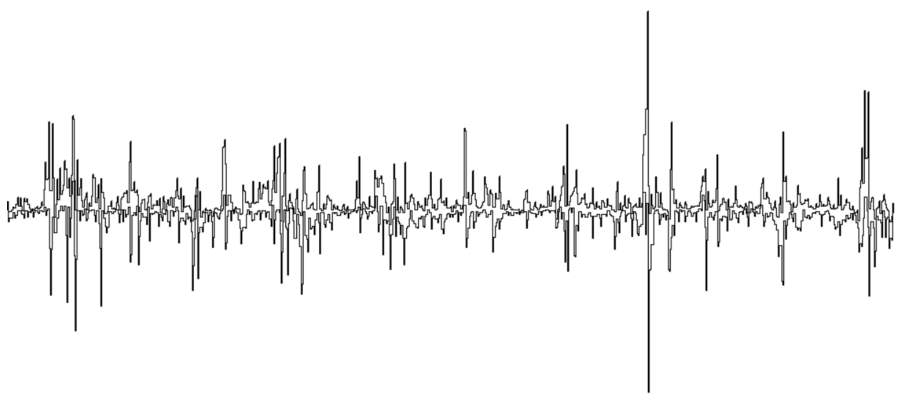

*Source: analysis performed in Software Image J*

As can be seen from Figure 4, the fractal dimension obtained by the box count method is very close to its theoretical value, therefore this approach will be used for further analysis.

**Fractal analysis of GEL/USD exchange rate and Bitcoin price**

Time series do not have perfect geometric characteristics. However, connecting dots on the time series graph form an object that we consider as fractal object.

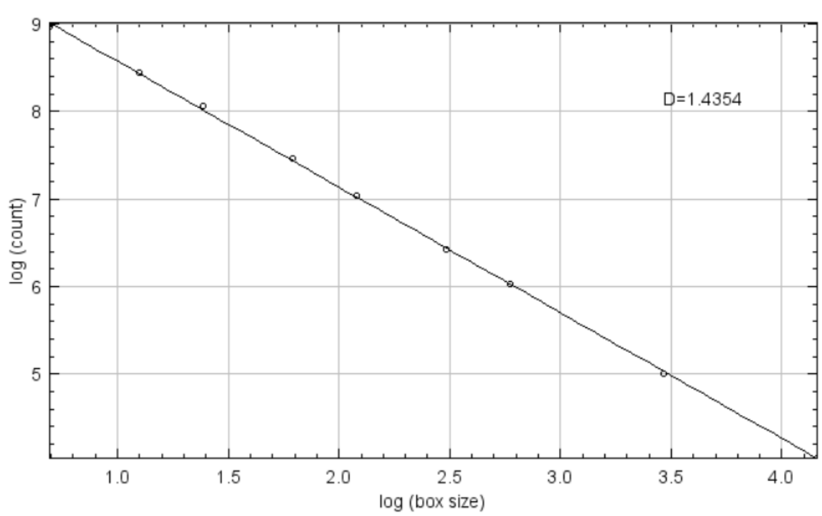
**Figure 5: Log change of the GEL/USD exchange rate as a fractal object**



*Source: National Bank of Georgia (image processed in Image j program)*

As can be seen in Figure 5, the exchange rate daily increment is represented as a geometric object/figure. Using the method described above, fractal dimension of the GEL/USD exchange rate is 1.43 (See Figure 6).

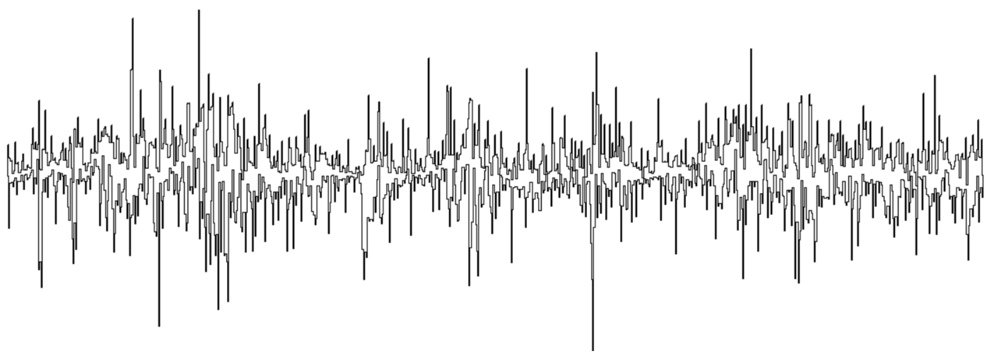
**Figure 6: Fractal dimension of GEL/USD exchange rate (log change)**




*Source: analysis performed in Software Image J*

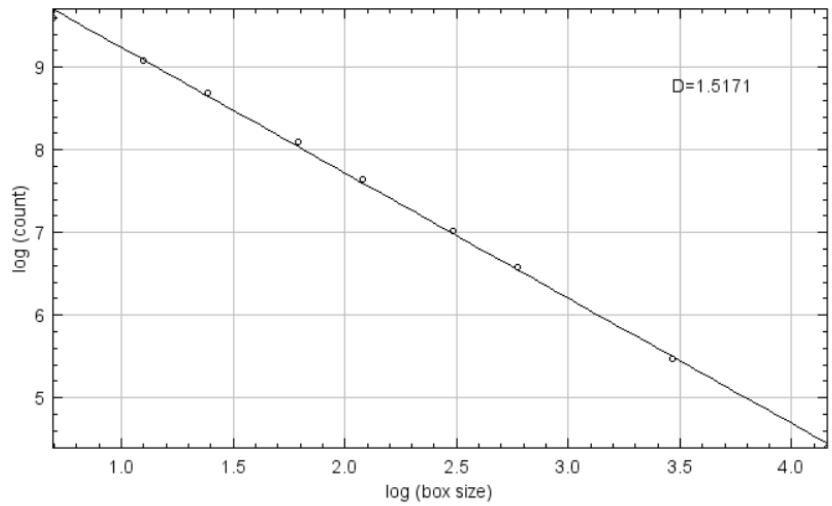
Similarly, we can consider a geometric representation of a time series of Bitcoin price changes and perform the same analysis.

**Figure 7: Fractal representation of Bitcoin price change**



*Source: National Bank of Georgia (image processed in Image j program)*

**Figure 8: Determining the Fractal Dimension of Bitcoin Price**




*Source: analysis performed in Software Image J*

As can be seen in Figures 7 and 8, the fractal time series of Bitcoin price change has a fractal dimension of 1.5171.

**Fractal dimension & Hurst Exponent**

There is the following relationship between the fractal dimension and the Hurst exponent: D = 2 – H (Hurst et al, 1965). This relationship will later on be used to define the correlation for the analyzed time series. Furthermore, Thanks to this connection, we can calculate the Hurst exponent through the fractal dimension and vice versa which will help us define validity of the approach.

The Hurst exponent was developed by Harold Edwin Hurst. The value of the exponent is between 0 – 1 and shows the presence or absence of long-term autocorrelation in the time series (Kantelhardt, 2008).

In particular, if the H value is between 0.5 and 1, this indicates a positive autocorrelation in the time series, and in the opposite case - a negative autocorrelation. When H = 0.5, then the time series is not characterized by autocorrelation (Kantelhardt, 2008).

Due to the fact that the above-mentioned relationship between the fractal dimension and the Hurst exponent may not be absolutely accurate (Taylor, 2018), in each case the H value will be calculated separately by means of the "normalized size" method (Rescaled Range Analysis).

This method is the transforming a time series of length M as follows (Kantelhardt, 2008):

$$\mathrm{N}i = \log\left(\frac{\mathrm{M}i+1}{\mathrm{M}i}\right) \quad , i = 1, 2, ..., (M - 1) \quad (3)$$

After that, the time interval of the time series will be divided into T number of sub-periods of length j. Denoting each interval by t and each element included in it by Nk,t gives the average value for each sub-period as follows:

$$\mathrm{e_t} = \frac{1}{\mathrm{j}}\sum_{\mathrm{k}=1}^{\mathrm{j}} \mathrm{N}k,t\,, \ \mathrm{k} = 1,2,,\mathrm{j} \quad (4)$$

In this way, we get the average value in each sub-period and calculate the average quadratic deviation for the same sub-periods:

$$S_{I_t} = \sqrt{\sum_1^{\mathrm{k}}(\mathrm{N}i,t - e_t)^2}\,, \ \mathrm{k} = 1,2,,\mathrm{j} \quad (5)$$

The maximum range in each sub-period will be calculated as follows:

$$\mathrm{R_{I_t}} = max(\mathrm{X_{k,t}}) - min(\mathrm{X_{k,t}}),\ 1 < k < j \quad (6)$$

After that, the average R/S value for each subgroup is calculated:

$$\mathrm{e_t} = \frac{1}{\mathrm{T}}\sum_{\mathrm{t}=1}^{\mathrm{T}} \frac{\mathrm{R_{I_t}}}{\mathrm{S_{I_t}}},\ \mathrm{k} = 1,2,,\mathrm{j} \quad (7)$$

As a result, by regressing the logarithmic value of the average R/S ratio obtained for each subgroup with the logarithm of the number of data included in each subgroup (log(n)), the value of H is obtained.

**Hurst exponent and fractal dimension**

Thanks to the relationship between the Hurst exponent and the fractal dimension, we can compare the results obtained by the two methods. Given the above methodology, the calculations for the USD/GEL exchange rate are as follows (see Table 1).

**Table 1: Calculation of the Hurst exponent of the USD/GEL exchange rate**

| Number of Groups | 2 | 4 | 8 | 16 | 32 | 64 | 128 | 256 | 512 |
|---|---|---|---|---|---|---|---|---|---|
| # of observations in sub groups | 1024 | 512 | 256 | 128 | 64 | 32 | 16 | 8 | 4 |
| average R/S | 21 | 21 | 23 | 25 | 32 | 40 | 49 | 48 | 65 |
| Log(R/S) | 3 | 3 | 3 | 3 | 3 | 4 | 4 | 4 | 4 |
| Log(N) | 7 | 6 | 6 | 5 | 4 | 3 | 3 | 2 | 1 |
| **Hurst exponent** | **0.680** | | | | | | | | |

*Source of Data: National Bank of Georgia*

As can be seen from Table 1, the Hurst exponent for the USD/GEL exchange rate is 0.68 units, which exceeds 0.5 and indicates a positive long-term correlation in the time series. By using the relationship D = 2 – H, we get that the fractal dimension determined in this way is 1.32 units, which is slightly different from the result obtained by the box counting method - 1.43.
In order to check the accuracy of the obtained indicator, we will conduct a t-test.

**Table 2: Testing Hurst's exponent for equality with 0.5**

| | |
|---|---|
| Hurst exponent | 0.68 |
| Standard Deviation | 0.14 |
| Threshold | 0.50 |
| t-test | 1.30 |
| Number of Observations | 9 |
| Degree of freedom | 7 |
| **P-value** | **0.236** |

*Source of Data: National Bank of Georgia*

As it can be seen from the obtained test results, we cannot reject the null hypothesis that the value of the obtained Hurst exponent is equal to 0.5. Which says that the data does not show long-term correlation.

Nevertheless, the value of 0.68 is greater than the value of 0.5, which is indicative of small autocorrelation. Also, as shown by the autocorrelation function, the time series is characterized by autocorrelation.

Similarly, let's conduct an analysis in the case of the price of Bitcoin. As in the previous case, the number of observation points in this case is 2048, which allows us to create 9 subgroups and perform the corresponding analysis.

**Table 3: Calculation of Hurst Exponent of Bitcoin Price**

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| Number of Groups | 2 | 4 | 8 | 16 | 32 | 64 | 128 | 256 | 512 |
| Number of observations in sub groups | 1024 | 512 | 256 | 128 | 64 | 32 | 16 | 8 | 4 |
| Average R/S | 15 | 16 | 16 | 17 | 17 | 18 | 20 | 20 | 23 |
| log(R/S) | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 |
| Log(N) | 7 | 6 | 6 | 5 | 4 | 3 | 3 | 2 | 1 |
| **Hurst exponent** | **0.575** | | | | | | | | |

*Source of Data: Yahoo Finance*

As can be seen from Table 3, the value of the Hurst exponent is 0.575, which is close to 0.5 telling us that there is no long-term autocorrelation process in the time series (ACF also confirmed the same). Similarly, we can also calculate the fractal dimension by using obtained value of Hurst exponent: 2-0.575 = 1.425, which is slightly different from the value obtained using the method of counting boxes – D= 1.517, thus fractal dimension seems to be right.

Both values indicate the absence of long-term autocorrelation in the time series, which means that the price of Bitcoin is solely formed by random events, which makes it difficult to predict.
To test the hypothesis that the mean of Hurst exponent is equal to 0.5, we again perform the t test.

**Table 2: Testing Hurst's exponent for equality with 0.5**

| | |
|---|---|
| Hurst exponent | 0.57 |
| Standard deviation | 0.10 |
| Threshold | 0.50 |
| t-test value | 0.76 |
| Number of Observations | 9 |
| Degree of freedom | 7 |
| **P-value** | **0.473** |

*Source of Data: Yahoo Finance*

T statistic proves the point and we cannot reject the hypothesis that Hurst's exponent equal to 0.5 with a fairly large P value.

**Conclusions**

Based on the analysis, we can say that price of Bitcoin is less dependent on previous values, it is formed by random factors, and therefore it is very difficult to predict. This result was fairly simple and expected, however fractal analysis proves to be effective in measuring the correlation.

In the case of the exchange rate, the time series is correlational object, the future values are highly dependent on past values. Although the T statistics for Hurst’s exponent suggested the opposite, still, the t-value was significantly higher than the threshold indicating the autocorrelation in the data. Nonetheless, using fractal dimensions as a measure for correlation in a time series data, might not be fully reliable method if used without combination with other methods and pre-knowledge about the data.

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